Numerical Integration

التكامل العددي

Numerical integration is used to obtain approximate values of definite integrals that can not be solved analytically.

It is a process of finding the numerical value of the definite integral:

$$I = \int_a^b f(x)\,dx$$

when $f(x)$ is:

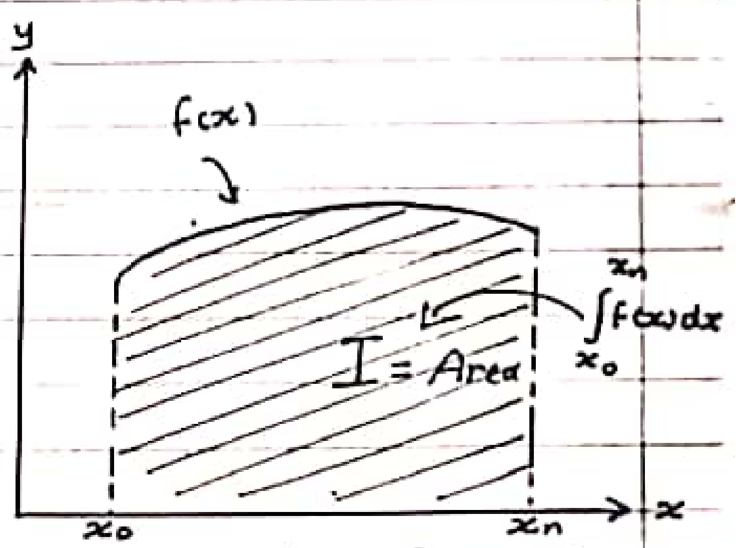
1. very difficult to integrate analytically.
2. is not known explicitly, but only a set of $x, f(x)$ values are known.

There are many methods to numerically integrate the function $f(x)$, or a set of points $x, f(x)$.

1. Rectangular method

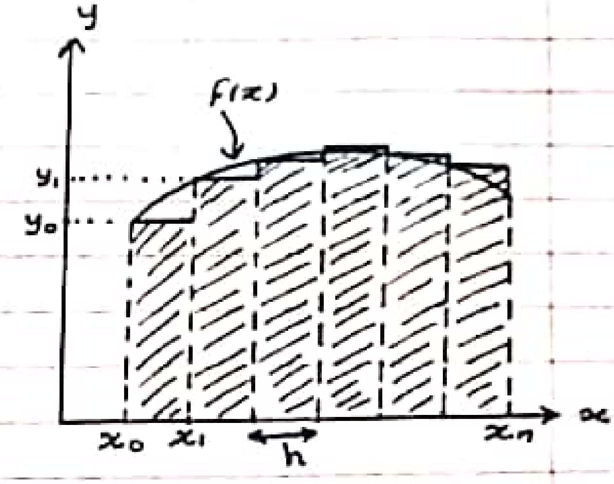
Recall the meaning of the integral:

$$I = \int_{x_0}^{x_n} f(x)\,dx$$



It is nothing but the area enclosed by the curve of the function and the x-axis.

To find the approximate value of the enclosed area we will devide the area to definite number of equi-distant rectangles as shown in the two figures

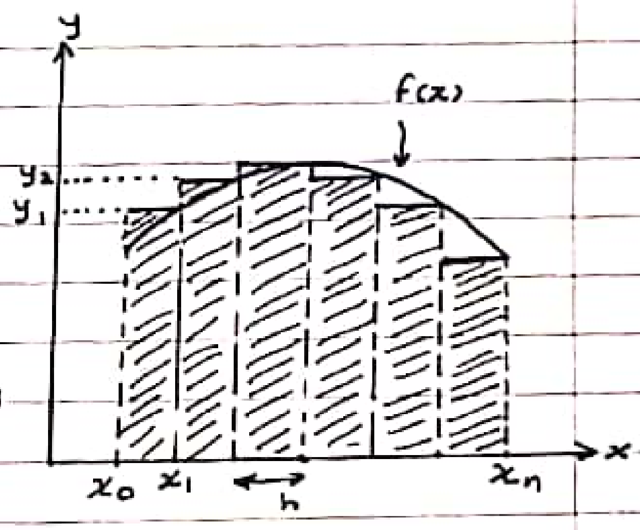


For the first figure :

The area of the rectangles are :

$h(y_0 + y_1 + y_2 + \cdots y_{n-1})$. This is an approximate value of the integral I :

$$I = \int_{x_0}^{x_n} f(x)\, dx$$



For the second figure:

The area of the rectangles are :

$h(y_1 + y_2 + y_3 + \cdots + y_n)$ which is an approximate value of the integral I.

Example :

Find the value of the integral

$$I = \int_1^2 x^2\, dx$$ By using the above two formulae of rectangular method.

$x_0 = 1 \quad x_n = 2 \quad f(x) = y = x^2$

We will use the value of $h = 0.2$

Construct the table :

	x	y	
x_0	1	1	y_0
x_1	1.2	1.44	y_1
x_2	1.4	1.96	y_2
x_3	1.6	2.56	y_3
x_4	1.8	3.24	y_4
x_5	2	4	y_5

From the first formula:

$$I_1 \simeq h(y_0 + y_1 + \cdots + y_4)$$

$$\simeq 0.2(1 + 1.44 + 1.96 + 2.56 + 3.24)$$

$$\simeq 2.04$$

From the second formula:

$$I_2 \simeq h(y_1 + y_2 + \cdots + y_5)$$

$$\simeq 0.2(1.44 + 1.96 + 2.56 + 3.24 + 4)$$

$$\simeq 2.64$$

The exact value of the integral is:

$$\int_1^2 x^2 dx = \left[\frac{x^3}{3}\right]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

$$= 2.333$$

To increase the accuracy we can increase the number of sub-intervals by decreasing the value of h :

Take $h = 0.1$

Construct the table :

	x	y	
x_0	1	1	y_0
x_1	1.1	1.21	y_1
x_2	1.2	1.44	y_2
x_3	1.3	1.69	y_3
x_4	1.4	1.96	y_4
x_5	1.5	2.25	y_5
x_6	1.6	2.56	y_6
x_7	1.7	2.89	y_7
x_8	1.8	3.24	y_8
x_9	1.9	3.61	y_9
x_{10}	2	4	y_{10}

By using the first formula:

$$I_1 \approx 0.1 (1 + 1.21 + 1.44 + 1.69 + 1.96 + 2.25 + 2.56 + 2.89 + 3.24 + 3.61)$$

$$\approx 2.185$$

By using the second formula :

$$I_2 \approx 0.1 (1.21 + 1.44 + 1.69 + 1.96 + 2.25 + 2.56 + 2.89 + 3.24 + 3.61 + 4)$$

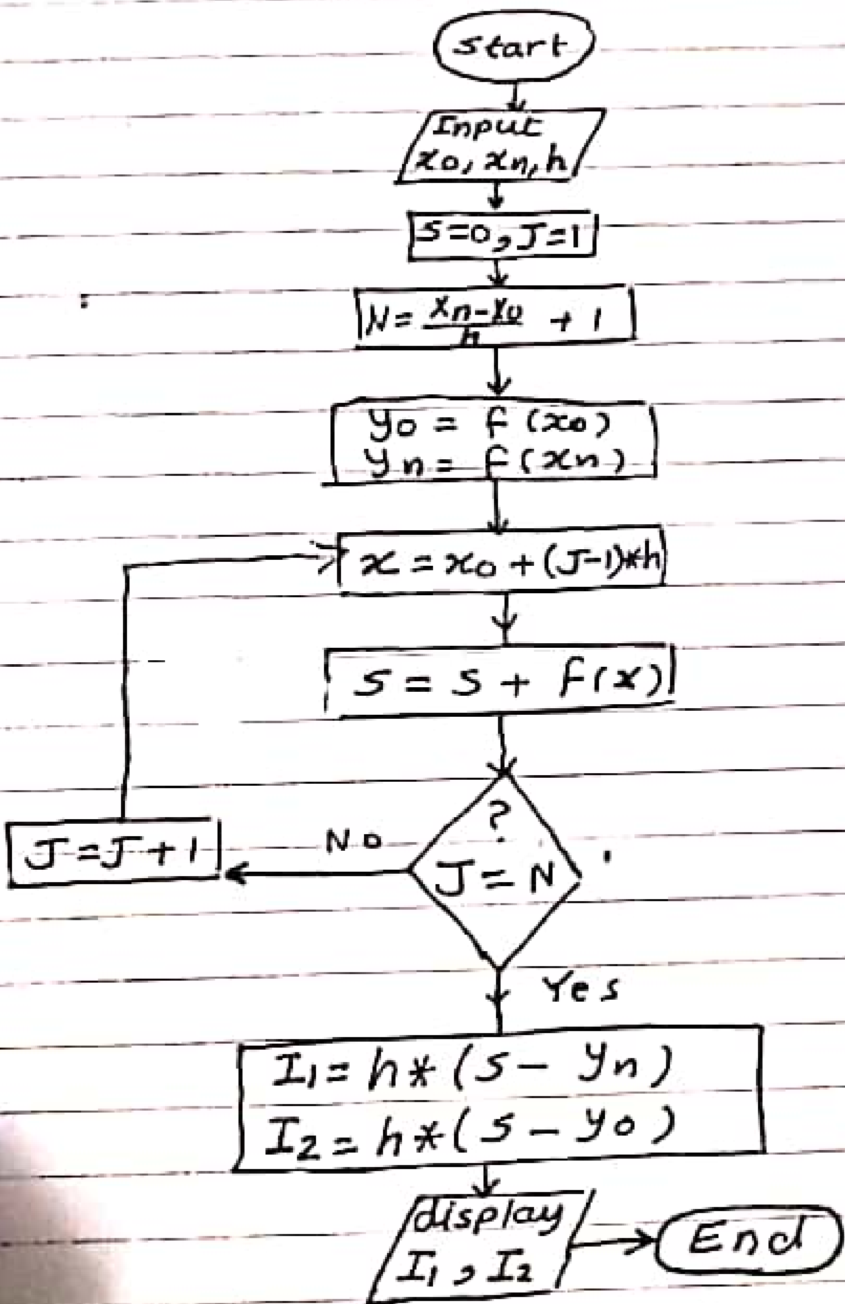
$\therefore\ I_2 \simeq 2.485$

For more accurate results :

1- Decrease the value of h

2- $I = (I_1 + I_2)/2$

Flow chart of rectangular method :



Home Work :

1. Calculate the value of the integral:

$$\int_0^1 \frac{x}{1+x}\,dx$$

use $h = 0.2$ Answer: $I \approx 0.305$

2. Calculate the value of the integral:

$$\int_0^1 \frac{dx}{1+x^2}$$ use $h=0.2$

Answer: $I \simeq 0.785$

3. Calculate the value of the integral:

$$\int_0^{0.6} e^x\,dx$$ use $h=0.1$

Answer: $I \simeq 0.822$